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AN EXCLUSIVE TRANSCENDENTAL EQUATION

$$\sqrt[3]{X^2+Y^2}+\sqrt[3]{Z^2+W^2}=(k^2+1)R^2$$

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Abstract

The transcendental equation $\sqrt[3]{X^2+Y^2}+\sqrt[3]{Z^2+W^2}=(k^2+1)R^2$ is analyzed for its non trivial-integral solutions. A few interesting relations between the solutions (X,Y,Z,W), special polygonal numbers and pyramidal numbers are presented.

Keywords: Transcendental equation, integral solutions, polygonal numbers and pyramidal numbers.
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Introduction

Diophantine equations are numerously rich because of its variety. In [1-7], the Diophantine equations for which we require integral solutions are algebraic equations with integer co-efficient. In [8-13], six different transcendental equations are studied for its non-trivial integral solutions. In this communication, the transcendental equation represented by $\sqrt[3]{X^2+Y^2}+\sqrt[3]{Z^2+W^2}=(k^2+1)R^2$ is analyzed for its non trivial-integral solutions. A few interesting relations between the solutions (X,Y,Z,W), special polygonal numbers and pyramidal numbers are presented.

Notations

Polygonal numbers	Notations for rank n	Definitions
Gnomonic	Gno_n	$2n-1$
Hexagonal	Hex_n	$n(2n-1)$
Stella Octangula	SO_n	$n(2n^2-1)$
Octahedral	OH_n	$\frac{1}{3}n(2n^2+1)$
Rhombic dodecagonal	RD_n	$(2n-1)(2n^2-2n+1)$
Pentagonal pyramidal	PP_n	$\frac{1}{2}n^2(n+1)$
Pronic	$\Pr o_n$	$n(n+1)$
Centered Square	CS_n	$2n^2-2n+1$
Triangular	T_n	$\frac{n(n+1)}{2}$

Tetrahedral	TH_n	$\frac{1}{6}n(n+1)(n+2)$
Centered m-gonal	$CT_{m,n}$	$\frac{m[n(n+1)+2]}{2}$
Decagonal	Dec_n	$n(4n-3)$

Explicit formulas for the above m-gonal numbers may be found in [14-16].

Method of Analysis

The transcendental equation to be solved is

$$\sqrt[3]{X^2+Y^2}+\sqrt[3]{Z^2+W^2}=(k^2+1)R^2 \qquad (1)$$

where $k \neq \{0\}$

Taking

$$\rho^3 = X^2+Y^2 \qquad (2)$$

The values of X and Y satisfying (2) are offered by

$$X = m(m^2+n^2) \qquad (3)$$

$$Y = n(m^2+n^2) \qquad (4)$$

Similarly, by choosing

$$\tau^3 = Z^2+W^2$$

the values of Z and W satisfying the above cubic equation are stated by

$$Z = m(m^2-3n^2) \qquad (5)$$

$$W = n(n^2-3m^2) \qquad (6)$$

On substituting (3),(4),(5) and (6) in (1),we obtain

$$2(m^2+n^2)=(k^2+1)R^2 \qquad (7)$$

Set

$$R = a^2+b^2$$

Applying unique factorization method, (7) can be written as

$$(1+i)(1-i)(m+ni)(m-ni)=(k+i)(k-i)(a^2-b^2+2abi)(a^2-b^2-2abi)$$

Thus,

$$(1+i)(m+ni)=(k+i)(a^2-b^2+2abi)$$

Equating real and imaginary parts, we search out

$$m+n = k(a^2-b^2+2kab)$$

$$m-n = k(a^2-b^2-2kab)$$

Solving the above two equations, we grasp

$$m=\frac{1}{2}\left[k(a^2-b^2+2ab)+a^2-b^2-2ab\right] \quad (8)$$

$$n=\frac{1}{2}\left[k(2ab-a^2+b^2)+a^2-b^2+2ab\right] \quad (9)$$

Here, the non-trivial integral solutions to (1) are analyzed when k is odd and k is even.

Case (i)

Consider $k=2\alpha+1$

This choice leads (8) and (9) to

$$m=\alpha(a^2-b^2+2ab)+a^2-b^2 \quad (10)$$

$$n=\alpha(2ab-a^2+b^2)+2ab \quad (11)$$

Substituting (10) and (11) in (3),(4),(5) and (6), the non-zero integral solutions to (1) are symbolized by

$$X=\left[\alpha(a^2-b^2+2ab)+a^2-b^2\right](a^2+b^2)^2(2\alpha^2+2\alpha+1)$$

$$Y=\left[\alpha(2ab-a^2+b^2)+2ab\right](a^2+b^2)^2(2\alpha^2+2\alpha+1)$$

$$Z=-2\alpha^3(a^6-6a^5b-15a^4b^2+20a^3b^3+15a^2b^4-6ab^5-b^6)+$$
$$3\alpha(a^6+6a^5b-15a^4b^2-20a^3b^3+15a^2b^4+6ab^5-b^6)+$$
$$12\alpha^2(3a^5b-10a^3b^3+3ab^5)+(a^6-15a^4b^2+15a^2b^4-b^6)$$

$$W=2\alpha^3(a^6+6a^5b-15a^4b^2-20a^3b^3+15a^2b^4+6ab^5-b^6)+$$
$$3\alpha(a^6-6a^5b-15a^4b^2+20a^3b^3+15a^2b^4-6ab^5-b^6)+$$
$$6\alpha^2(a^6-15a^4b^2+15a^2b^4-b^6)+(-6a^5b+20a^3b^3-6ab^5)$$

To find the relations among the solutions, we consider the choice $a=b$

Hence,

$$X=8a^6(2\alpha^3+2\alpha^2+\alpha)$$

$$Y=8a^6(\alpha+1)(2\alpha^2+2\alpha+1)$$

$$Z=-8a^6(2\alpha^3+6\alpha^2+3\alpha)$$

$$W = -8a^6(2\alpha^3 - 3\alpha - 1)$$

A few interesting relations among the solutions are expressed below:

1. $\dfrac{Y - X}{8a^6} = CT_{4,\alpha}$

2. $\dfrac{X}{8a^6} - 12TH_\alpha + 10Dec_\alpha \equiv 0(\bmod 6)$

3. $\dfrac{X + Y}{8a^6} = PP_{2\alpha} + 8T_\alpha + 1$

4. $X + W = 8a^6(4T_\alpha + Gno_{\alpha+1})$

5. $W + 8a^6(SO_\alpha + Gno_{\alpha+1}) = 0$

6. $Z + 8a^6(2PP_\alpha + 6T_\alpha) - 8\alpha^2 a^6 = 0$

7. $X + Y + Z + W = 16a^6 Gno_{\alpha+1}$

8. Each of the following expressions provides a nasty number

   (i) $3(2X + Z + W + 16a^6 \Pr o_{\alpha-1})$

   (ii) $3(X - 24a^6 OH_\alpha)$

   (iii) $3(\alpha+1)[Y - 8a^6(4PP_\alpha + CS_\alpha)]$

Case (ii)

Choosing $k = 2\alpha$ in (8) and (9), it reduces to

$$m = \left[\alpha(a^2 - b^2 + 2ab) + \frac{a^2 - b^2 - 2ab}{2}\right]$$

$$n = \left[\alpha(2ab - a^2 + b^2) + \frac{a^2 - b^2 + 2ab}{2}\right]$$

Since our interest centers on finding integral solution, we observe that m and n are integers for $a = 2A$ and $b = 2B$

Thus,

$$m = 4\alpha(A^2 - B^2 + 2AB) + 2(A^2 - B^2 - 2AB)$$

$$n = 4\alpha(2AB - A^2 + B^2) + 2(A^2 - B^2 + 2AB)$$

In view of (3),(4),(5) and (5),the non-zero integer solutions fulfilling (1) are exhibited by

$$X = 8\left[4\alpha(A^2 - B^2 + 2AB) + 2(A^2 - B^2 - 2AB)\right](A^2 + B^2)(4\alpha^2 + 1)$$

$$Y = 8\left[4\alpha(2AB - A^2 + B^2) + 2(A^2 - B^2 + 2AB)\right](A^2 + B^2)(4\alpha^2 + 1)$$

$$Z = \left[4\alpha(A^2 - B^2 + 2AB) + 2(A^2 - B^2 - 2AB\right]\times \left\{4(4\alpha^2 - 3)(A^2 - B^2 + 2AB)^2 - 4(12\alpha^2 + 1)(A^2 - B^2 - 2AB)^2 + 64\alpha\left[(A^2 - B^2)^2 - 4A^2B^2\right]\right\}$$

$$W = \left[4\alpha(2AB - A^2 + B^2) + 2(A^2 - B^2 + 2AB\right]\times \left\{4(4\alpha^2 - 3)(A^2 - B^2 - 2AB)^2 - 4(12\alpha^2 + 1)(A^2 - B^2 + 2AB)^2 - 64\alpha\left[(A^2 - B^2)^2 - 4A^2B^2\right]\right\}$$

To obtain some properties between the solutions we select $A = B$

Thus,

$$X = 128A^6(8\alpha^3 - 4\alpha^2 + 2\alpha - 1)$$

$$Y = 128A^6(8\alpha^3 + 4\alpha^2 + 2\alpha + 1)$$

$$Z = -128A^6(2\alpha - 1)(4\alpha^2 + 8\alpha + 1)$$

$$W = -128A^6(2\alpha + 1)(4\alpha^2 - 8\alpha + 1)$$

A few relations among the solutions are furnished below:

1. $X = 128A^6(RD_\alpha + 4T_\alpha \times Gno_\alpha$
2. $Y = 128A^6(RD_{\alpha+1} + 4T_{\alpha-1} \times Gno_{\alpha+1}$
3. $Y = 128A^6(2PP_{2\alpha} + Gno_{\alpha+1})$
4. $X + Z + 1024A^6 Hex_\alpha = 0$
5. $Y + W = 1024A^6 Hex_{\alpha+1} - 1024Gno_{\alpha+1}$
6. Each of the following expressions characterizes a nasty number

(i) $X + Y - 128A^6(3OH_{2\alpha} + Gno_\alpha 1)$

(ii) $6\dfrac{X+Z}{(8T_{\alpha}+1)Gno_{\alpha}}$

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